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EXACT ANALYTICAL SOLUTION OF THE KIRSCH PROBLEM WITHIN THE FRAMEWORK OF THE COSSERAT CONTINUUM AND PSEUDOCONTINUUM

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The Kirsch problem of one-sided tension of a plate with a circular hole is considered within the framework of the nonsymmetric theory of elasticity under the assumption that material deformation is described not only by the displacement vector but also by the rotation vector. The general analytical solution of this problem is expressed in terms of the Bessel functions. The resulting solution is compared with the corresponding solutions for a symmetric medium and Cosserat pseudomedium. A macroparameter characterizing the distortion of the boundary of the circular hole upon deformation is introduced.

Introduction. In 1910, the Cosserat brothers proposed a model of a medium according to which the deformation of the medium is governed not only by the displacement vector $\boldsymbol{u}$ but also by the rotation vector $\boldsymbol{\omega}$, which depend on coordinates and time. The medium modeled in this manner is called the Cosserat medium, and the theory is called the moment or nonsymmetric theory of elasticity.

This theory was independently developed by several researchers [1–5] in the 1960s and 1970s. At that time, the first analytical solutions of two-dimensional problems were obtained within the framework of the moment theory. But most of the exact solutions were based on the simplifying assumption

$$\boldsymbol{\omega} = \frac{1}{2}\,\mathrm{rot}\,\boldsymbol{u}, \qquad (1)$$

which is called the constrained rotation or Cosserat pseudomedium. In this variant of the moment theory of elasticity, the number of physical constants of an isotropic elastic body is reduced from six to four [6]. Moreover, the structure of the resulting equations [1] is such that if, in particular, displacements are specified at the surface of an elastic body, the normal component of the rotation vector depends on the displacement vector.

The aim of this paper is to construct and analyze the solution of the Kirsch problem of uniaxial tension of an infinite plate with a central circular hole for a nonsymmetric medium. We perform a parametric analysis of the exact solution and show that it can be used in experiments to identify the physicomechanical parameters of the Cosserat continuum.

Kirsch was first to solve this problem by the methods of the classical theory of elasticity. Later, Muskhelishvili [7] solved the problem by another method. In [6, 8, 9], this solution was generalized to the Cosserat pseudomedium. Pal'mov [4] studied the stress concentration near the circular hole within the framework of the nonsymmetric theory of elasticity.

It should be noted that the solution given in [4] does not allow one to obtain full information on the stress–strain state in the neighborhood of the circular hole, in particular, to determine the degree of distortion of the hole due to deformation.

In the present paper, the exact analytical solution of the Kirsch problem is obtained within the framework of the general moment theory of elasticity. The solution is written in a dimensionless form, which enables one to perform its parametric analysis.

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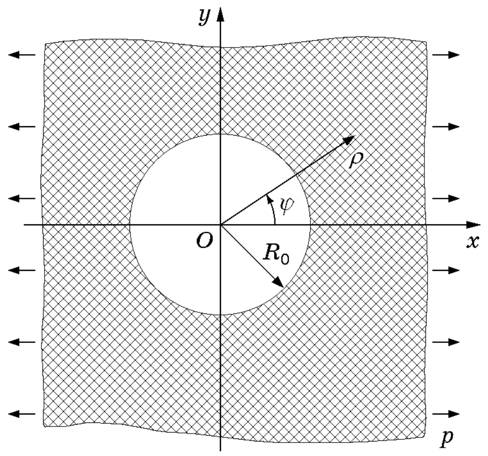

Fig. 1

1. Formulation of the Problem. We write the basic relations of the moment theory of elasticity [1]:
— Equations of equilibrium

$$\nabla \cdot \tilde{\sigma} + \boldsymbol{X} = \boldsymbol{0}, \qquad \tilde{\sigma}^{\rm t} : \tilde{E} + \nabla \cdot \tilde{\mu} + \boldsymbol{Y} = \boldsymbol{0}; \tag{1.1}$$

— Geometrical relations

$$\tilde{\gamma} = \nabla \boldsymbol{u} - \tilde{E} \cdot \boldsymbol{\omega}, \qquad \tilde{\chi} = \nabla \boldsymbol{\omega}; \tag{1.2}$$

— Physical relations

$$\tilde{\sigma} = 2\mu \tilde{\gamma}^{(S)} + 2\alpha \tilde{\gamma}^{(A)} + \lambda I_1(\tilde{\gamma})\tilde{e}, \qquad \tilde{\mu} = 2\gamma \tilde{\chi}^{(S)} + 2\varepsilon \tilde{\chi}^{(A)} + \beta I_1(\tilde{\chi})\tilde{e}. \tag{1.3}$$

With allowance for relations (1.1)–(1.3), the equations of equilibrium for the displacement vector $\boldsymbol{u}$ and the rotation vector $\boldsymbol{\omega}$ are written in the form

$$\begin{gathered}(2\mu + \lambda)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} - (\mu + \alpha)\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{u} + 2\alpha\,\mathrm{rot}\,\boldsymbol{\omega} + \boldsymbol{X} = \boldsymbol{0},\\ (\beta + 2\gamma)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega} - (\gamma + \varepsilon)\,\mathrm{rot}\,\mathrm{rot}\,\boldsymbol{\omega} + 2\alpha\,\mathrm{rot}\,\boldsymbol{u} - 4\alpha\boldsymbol{\omega} + \boldsymbol{Y} = \boldsymbol{0}.\end{gathered} \tag{1.4}$$

In (1.1)–(1.4), $\boldsymbol{X}$ is the vector of mass forces, $\boldsymbol{Y}$ is the vector of mass moments, $\boldsymbol{u}$ is the displacement vector, $\boldsymbol{\omega}$ is the rotation vector, $\tilde{\gamma}$ and $\tilde{\chi}$ are the strain and flexure–torsion tensors, respectively, $\tilde{\sigma}$ and $\tilde{\mu}$ are the stress and couple-stress tensors, respectively, μ and λ are the Lamé constants, α, β, γ, and ε are the physical constants of the material within the framework of the moment theory of elasticity, $\tilde{E}$ is the Levi-Civita tensor of the third rank, $(\,\cdot\,)^{(S)}$ is the operation of symmetrization, $(\,\cdot\,)^{(A)}$ is the operation of alternation, $\nabla(\,\cdot\,)$ is the nabla operator, $I_1(\,\cdot\,)$ is the first invariant of the tensor, and $\tilde{e}$ is the unit tensor [10]. In contrast to the classical theory, the tensors $\tilde{\gamma}$ and $\tilde{\sigma}$ are nonsymmetric.

We also consider the simplified theory [1] in which the rotation vector is assumed to satisfy relation (1). Below, we call a medium with this property the Cosserat pseudomedium.

The physical relations of the Cosserat pseudomedium have the form

$$\tilde{\sigma} = 2\mu \tilde{\gamma}^{(A)} + \lambda I_1(\tilde{\gamma})\tilde{e} - (1/2)\nabla \cdot \tilde{\mu} \cdot \tilde{E}, \qquad \tilde{\mu} = 2\gamma \tilde{\chi}^{(S)} + 2\varepsilon \tilde{\chi}^{(A)} + \beta I_1(\tilde{\chi})\tilde{e}. \tag{1.5}$$

As in the complete moment formulation, the components of the strain and flexure–torsion tensors are determined with the use of relation (1.2). With allowance for (1.5), however, the equations of equilibrium of the Cosserat pseudomedium differ from (1.4):

$$\mu\nabla^2\boldsymbol{u} + (\mu + \lambda)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{u} + (1/4)(\gamma + \varepsilon)\,\mathrm{rot}\,\mathrm{rot}\,\nabla^2\boldsymbol{u} + \boldsymbol{X} = \boldsymbol{0}. \tag{1.6}$$

We consider the problem of one-sided tension of a plate with a circular hole. Let the boundary of the hole be stress-free and tensile forces of a constant intensity p act in the Ox direction at infinity (Fig. 1).

By virtue of symmetry, we seek the solution of the problem in a cylindrical system of coordinates (ρ, φ, z) as an expansion into harmonics

$$\boldsymbol{u}(\rho,\varphi) = \{F(\rho) + U(\rho)\cos 2\varphi, V(\rho)\sin 2\varphi, 0\}, \quad \boldsymbol{\omega}(\rho,\varphi) = \{0, 0, \omega(\rho)\sin 2\varphi\}. \tag{1.7}$$

The boundary conditions at infinity and at the stress-free contour of the hole have the form

$$\boldsymbol{n}_1 \cdot \tilde{\sigma}\Big|_{\rho=R_0} = \boldsymbol{0}, \qquad \boldsymbol{n}_1 \cdot \tilde{\mu}\Big|_{\rho=R_0} = \boldsymbol{0}, \qquad \boldsymbol{n}_2 \cdot \tilde{\sigma}\Big|_{\rho\to\infty} = \boldsymbol{p}, \qquad \boldsymbol{n}_2 \cdot \tilde{\mu}\Big|_{\rho\to\infty} = \boldsymbol{0},$$

where $\boldsymbol{n}_1 = \{-1, 0\}$ is the external normal to the circle of radius R_0 and $\boldsymbol{n}_2 = \{1, 0\}$ is the external normal to the circle of radius $R_1 \to \infty$.

In cylindrical coordinates, the components p_ρ and p_φ of the vector $\boldsymbol{p}$ are written in the forms $p_\rho = p(1 + \cos 2\varphi)/2$ and $p_\varphi = -(p\sin 2\varphi)/2$. Passing to the components of the stress and moment tensors, we write the boundary conditions as follows:

$$\begin{aligned} \sigma_{\rho\rho}\Big|_{\rho=R_0} &= 0, & \sigma_{\rho\varphi}\Big|_{\rho=R_0} &= 0, & \mu_{\rho z}\Big|_{\rho=R_0} &= 0, \\ \sigma_{\rho\rho}\Big|_{\rho\to\infty} &= p_\rho, & \sigma_{\rho\varphi}\Big|_{\rho\to\infty} &= p_\varphi, & \mu_{\rho z}\Big|_{\rho\to\infty} &= 0. \end{aligned} \tag{1.8}$$

Thus, the solution of the problem reduces to determining four functions $F(\rho)$, $U(\rho)$, $V(\rho)$, and $\omega(\rho)$ for the Cosserat medium and three functions $F(\rho)$, $U(\rho)$, and $V(\rho)$ for the Cosserat pseudomedium.

2. Analytical Solution of the Equation of Equilibrium. Substituting the displacement and rotation vectors (1.7) into (1.4), we obtain equations of equilibrium in the form of a system of the second-order linear differential equations for the functions $F(\rho)$, $U(\rho)$, $V(\rho)$, and $\omega(\rho)$:

$$\begin{gathered} \frac{d^2}{d\rho^2} U(\rho) = -\frac{1}{\rho}\frac{d}{d\rho} U(\rho) - \frac{A_{11}}{\rho^2} U(\rho) - \frac{A_{12}}{\rho}\frac{d}{d\rho} V(\rho) - \frac{A_{13}}{\rho^2} V(\rho) + \frac{A_{14}}{\rho}\omega(\rho), \\ \frac{d^2}{d\rho^2} V(\rho) = -\frac{1}{\rho}\frac{d}{d\rho} V(\rho) - \frac{A_{21}}{\rho^2} V(\rho) - \frac{A_{22}}{\rho}\frac{d}{d\rho} U(\rho) - \frac{A_{23}}{\rho^2} U(\rho) - A_{24}\frac{d}{d\rho}\omega(\rho), \\ \frac{d^2}{d\rho^2}\omega(\rho) = -\frac{1}{\rho}\frac{d}{d\rho}\omega(\rho) + 4A_{31}\omega(\rho) + \frac{4}{\rho^2}\omega(\rho) - 2A_{31}\frac{d}{d\rho} V(\rho) - \frac{2A_{31}}{\rho} V(\rho) - \frac{4A_{31}}{\rho} U(\rho), \\ \frac{d^2}{d\rho^2} F(\rho) = -\frac{1}{\rho}\frac{d}{d\rho} F(\rho) + \frac{1}{\rho^2} F(\rho). \end{gathered} \tag{2.1}$$

Here

$$A_{11} = -\frac{\lambda + 6\mu + 4\alpha}{\lambda + 2\mu}, \qquad A_{12} = -2\,\frac{\alpha - \lambda - \mu}{\lambda + 2\mu}, \qquad A_{13} = -2\,\frac{\lambda + 3\mu + \alpha}{\lambda + 2\mu},$$

$$A_{14} = -\frac{4\alpha}{\lambda + 2\mu}, \qquad A_{21} = -\frac{4\lambda + 9\mu + \alpha}{\alpha + \mu}, \qquad A_{22} = 2\,\frac{\alpha - \lambda - \mu}{\alpha + \mu},$$

$$A_{23} = -2\,\frac{\lambda + 3\mu + \alpha}{\alpha + \mu}, \qquad A_{24} = -\frac{2\alpha}{\alpha + \mu}, \qquad A_{31} = \frac{\alpha}{\gamma + \varepsilon}.$$

Substituting the displacement and rotation vectors (1.7) into (1.6), we obtain equations of equilibrium for the Cosserat pseudomedium in the form of a system of linear differential equations for the functions $F(\rho)$, $U(\rho)$, and $V(\rho)$:

$$\begin{gathered} \frac{d^3}{d\rho^3} V(\rho) = -\frac{2}{\rho}\frac{d^2}{d\rho^2} V(\rho) - \Big(4B_1 - \frac{5}{\rho^2}\Big)\frac{d}{d\rho} V(\rho) + \Big(4\,\frac{B_3}{\rho} + \frac{3}{\rho^3}\Big) V(\rho) \\ - \Big(2B_2\rho + \frac{2}{\rho}\Big)\frac{d^2}{d\rho^2} U(\rho) - \Big(2B_2 - \frac{2}{\rho^2}\Big)\frac{d}{d\rho} U(\rho) + \Big(2\,\frac{B_4}{\rho} + \frac{6}{\rho^3}\Big) U(\rho), \\ \frac{d^4}{d\rho^4} V(\rho) = -\frac{2}{\rho}\frac{d^3}{d\rho^3} V(\rho) + \Big(4B_5 + \frac{7}{\rho^2}\Big)\frac{d^2}{d\rho^2} V(\rho) \\ + \Big(4\,\frac{B_5}{\rho} - \frac{7}{\rho^3}\Big)\frac{d}{d\rho} V(\rho) - \Big(4\,\frac{B_6}{\rho^2} + \frac{9}{\rho^4}\Big) V(\rho) - \frac{2}{\rho}\frac{d^3}{d\rho^3} U(\rho) \end{gathered} \tag{2.2}$$

$$+\frac{4}{\rho^2}\frac{d^2}{d\rho^2}U(\rho)-\Big(8\frac{B_1}{\rho}-\frac{2}{\rho^3}\Big)\frac{d}{d\rho}U(\rho)-\Big(8\frac{B_3}{\rho^2}+\frac{18}{\rho^4}\Big)U(\rho),$$

$$\frac{d^2}{d\rho^2}F(\rho)=-\frac{1}{\rho}\frac{d}{d\rho}F(\rho)+\frac{1}{\rho^2}F(\rho).$$

Here

$$B_1=\frac{\lambda+\mu}{\gamma+\varepsilon},\quad B_2=\frac{\lambda+2\mu}{\gamma+\varepsilon},\quad B_3=\frac{\lambda+3\mu}{\gamma+\varepsilon},\quad B_4=\frac{\lambda+6\mu}{\gamma+\varepsilon},\quad B_5=\frac{\mu}{\gamma+\varepsilon},\quad B_6=\frac{4\lambda+9\mu}{\gamma+\varepsilon}.$$

The general solution of system (2.1) has the form

$$F(\rho)=C_1\rho+C_2\frac{1}{\rho},\quad U(\rho)=\sum_{i=3}^{8}C_iU_i(\rho),\quad V(\rho)=\sum_{i=3}^{8}C_iV_i(\rho),\quad \omega(\rho)=\sum_{i=3}^{8}C_i\omega_i(\rho),$$

where $U_i(\rho)$, $V_i(\rho)$, $\omega_i(\rho)$ ($i=3,\dots,8$) are particular solutions of system (2.1) and C_i are arbitrary constants determined from the boundary conditions (1.8).

As $U_3(x)$, $V_3(x)$, $U_4(x)$, $V_4(x)$, $U_5(x)$, $V_5(x)$, $U_6(x)$, and $V_6(x)$, we use particular solutions corresponding to the classical theory of elasticity, and $\omega_3(x)$, $\omega_4(x)$, $\omega_5(x)$, and $\omega_6(x)$ are obtained with the use of relation (1). This approach is substantiated in [1]. The remaining particular solutions, which we call the moment solutions, are found directly from system (2.1).

For convenient analysis of the resulting solution, we nondimensionalize all quantities. In this case, the dimensionless parameters ρ, u_ρ, u_φ, ω_z, γ_{ij}, χ_{ij}, σ_{ij}, μ_{ij}, and p are related to the dimensional parameters $\hat\rho$, $\hat u_\rho$, $\hat u_\varphi$, $\hat\omega_z$, $\hat\gamma_{ij}$, $\hat\chi_{ij}$, $\hat\sigma_{ij}$, $\hat\mu_{ij}$, and $\hat p$ by the formulas

$$\begin{gathered}\hat\rho=R_0\rho,\qquad \hat u_i=R_0u_i,\qquad \hat\sigma_{ij}=\mu\sigma_{ij},\qquad \hat p=\mu p,\\ \hat\mu_{ij}=R_0\mu\mu_{ij},\qquad \hat\gamma_{ij}=\gamma_{ij},\qquad \hat\chi_{ij}=\chi_{ij}/R_0.\end{gathered}\tag{2.3}$$

Furthermore, we introduce three dimensionless parameters one of which depends on the characteristic dimension R_0:

$$A=R_0\sqrt{\frac{\mu}{B(\gamma+\varepsilon)}},\qquad B=\frac{\alpha+\mu}{\alpha},\qquad C=\frac{\gamma-\varepsilon}{\gamma+\varepsilon}.\tag{2.4}$$

Using (2.3) and (2.4), we obtain the general solution for the components of the displacement and rotation vectors, and stress and couple-stress tensors in the dimensionless form:

$$u_\rho(\rho,\varphi)=C_1\rho+\frac{C_2}{\rho}+\Big(\frac{C_3}{\rho^3}+\frac{C_4}{\rho}+C_5\rho+C_6\rho^3+C_7U_7(\rho)+C_8U_8(\rho)\Big)\cos 2\varphi,$$

$$u_\varphi(\rho,\varphi)=\Big(\frac{C_3}{\rho^3}-C_4\frac{æ-1}{(æ+1)\rho}-C_5\rho-C_6\frac{æ+3}{æ-3}\rho^3+C_7V_7(\rho)+C_8V_8(\rho)\Big)\sin 2\varphi,$$

$$\omega_z(\rho,\varphi)=\Big(\frac{C_4}{\rho^2}-C_6\frac{3(æ+1)}{3-æ}\rho^2+C_7\omega_7(\rho)+C_8\omega_8(\rho)\Big)\sin 2\varphi,$$

$$\begin{gathered}\sigma_{\rho\rho}(\rho,\varphi)=C_1\frac{4}{æ-1}-\frac{2C_2}{\rho^2}+\Big(-C_3\frac{6}{\rho^4}-C_4\frac{8}{(æ+1)\rho^2}+2C_5+C_7S^{(7)}_{\rho\rho}(\rho)+C_8S^{(8)}_{\rho\rho}(\rho)\Big)\cos 2\varphi,\\ \sigma_{\rho\varphi}(\rho,\varphi)=\Big(-C_3\frac{6}{\rho^4}-C_4\frac{4}{(æ+1)\rho^2}-2C_5-C_6\frac{12}{3-æ}\rho^2+C_7S^{(7)}_{\rho\varphi}(\rho)+C_8S^{(8)}_{\rho\varphi}(\rho)\Big)\sin 2\varphi,\\ \sigma_{\varphi\rho}(\rho,\varphi)=\Big(-C_3\frac{6}{\rho^4}-C_4\frac{4}{(æ+1)\rho^2}-2C_5-C_6\frac{12}{3-æ}\rho^2+C_7S^{(7)}_{\varphi\rho}(\rho)+C_8S^{(8)}_{\varphi\rho}(\rho)\Big)\sin 2\varphi,\\ \sigma_{\varphi\varphi}(\rho,\varphi)=C_1\frac{4}{æ-1}+\frac{2C_2}{\rho^2}+\Big(C_3\frac{6}{\rho^4}-2C_5-C_6\frac{24}{3-æ}\rho^2+C_7S^{(7)}_{\varphi\varphi}(\rho)+C_8S^{(8)}_{\varphi\varphi}(\rho)\Big)\cos 2\varphi,\\ \mu_{\rho z}(\rho,\varphi)=\Big(-C_4\frac{2}{A^2B\rho^3}-C_6\frac{6(æ+1)}{(3-æ)A^2B}\rho+C_7M^{(7)}_{\rho z}(\rho)+C_8M^{(8)}_{\rho z}(\rho)\Big)\sin 2\varphi,\end{gathered}\tag{2.5}$$

$$\mu_{\varphi z}(\rho,\varphi) = \Big(C_4\,\frac{2}{A^2B\rho^3} - C_6\,\frac{6(æ+1)}{(3-æ)A^2B}\,\rho + C_7M^{(7)}_{\varphi z}(\rho) + C_8M^{(8)}_{\varphi z}(\rho)\Big)\cos 2\varphi,$$

$$\mu_{z\rho}(\rho,\varphi) = C\mu_{\rho z}(\rho,\varphi), \quad \mu_{z\varphi}(\rho,\varphi) = C\mu_{\varphi z}(\rho,\varphi), \quad D = |u_\rho(R_0,0)/u_\rho(R_0,\pi/2)|.$$

Here $æ = (3\mu+\lambda)/(\mu+\lambda)$ and D is a macroquantity that characterizes the distortion of the contour of the circular hole caused by uniaxial load (this quantity can be measured in experiments).

The functions ρ of the constants C_7 and C_8 in (2.5) are determined by the corresponding particular moment solutions and have the form

$$U_7(\rho) = \frac{1}{A^2\rho}\,I_0(2A\rho) - \frac{1}{A^3\rho^2}\,I_1(2A\rho), \qquad U_8(\rho) = \frac{1}{A^2\rho}\,K_0(2A\rho) + \frac{1}{A^3\rho^2}\,K_1(2A\rho),$$

$$V_7(\rho) = \frac{1}{A^2\rho}\,I_0(2A\rho) - \frac{1+A^2\rho^2}{A^3\rho^2}\,I_1(2A\rho), \qquad V_8(\rho) = \frac{1}{A^2\rho}\,K_0(2A\rho) + \frac{1+A^2\rho^2}{A^3\rho^2}\,K_1(2A\rho),$$

$$\omega_7(\rho) = -BI_0(2A\rho) + \frac{B}{A\rho}\,I_1(2A\rho), \qquad \omega_8(\rho) = -BK_0(2A\rho) - \frac{B}{A\rho}\,K_1(2A\rho),$$

$$S^{(7)}_{\rho\rho}(\rho) = -\frac{6}{A^2\rho^2}\,I_0(2A\rho) + \frac{6+4A^2\rho^2}{A^3\rho^3}\,I_1(2A\rho),$$

$$S^{(8)}_{\rho\rho}(\rho) = -\frac{6}{A^2\rho^2}\,K_0(2A\rho) - \frac{6+4A^2\rho^2}{A^3\rho^3}\,K_1(2A\rho),$$

$$S^{(7)}_{\rho\varphi}(\rho) = -\frac{6}{A^2\rho^2}\,I_0(2A\rho) + \frac{6+2A^2\rho^2}{A^3\rho^3}\,I_1(2A\rho),$$

$$S^{(8)}_{\rho\varphi}(\rho) = -\frac{6}{A^2\rho^2}\,K_0(2A\rho) - \frac{6+2A^2\rho^2}{A^3\rho^3}\,K_1(2A\rho), \tag{2.6}$$

$$S^{(7)}_{\varphi\rho}(\rho) = -\frac{6+4A^2\rho^2}{A^2\rho^2}\,I_0(2A\rho) + \frac{6+6A^2\rho^2}{A^3\rho^3}\,I_1(2A\rho),$$

$$S^{(8)}_{\varphi\rho}(\rho) = -\frac{6+4A^2\rho^2}{A^2\rho^2}\,K_0(2A\rho) - \frac{6+6A^2\rho^2}{A^3\rho^3}\,K_1(2A\rho),$$

$$S^{(7)}_{\varphi\varphi}(\rho) = \frac{6}{A^2\rho^2}\,I_0(2A\rho) - \frac{6+4A^2\rho^2}{A^3\rho^3}\,I_1(2A\rho), \quad S^{(8)}_{\varphi\varphi}(\rho) = \frac{6}{A^2\rho^2}\,K_0(2A\rho) + \frac{6+4A^2\rho^2}{A^3\rho^3}\,K_1(2A\rho),$$

$$M^{(7)}_{\rho z}(\rho) = \frac{2}{A^2\rho}\,I_0(2A\rho) - \frac{2+2A^2\rho^2}{A^3\rho^2}\,I_1(2A\rho), \quad M^{(8)}_{\rho z}(\rho) = \frac{2}{A^2\rho}\,K_0(2A\rho) + \frac{2+2A^2\rho^2}{A^3\rho^2}\,K_1(2A\rho),$$

$$M^{(7)}_{\varphi z}(\rho) = -\frac{2}{A^2\rho}\,I_0(2A\rho) + \frac{2}{A^3\rho^2}\,I_1(2A\rho), \quad M^{(8)}_{\varphi z}(\rho) = -\frac{2}{A^2\rho}\,K_0(2A\rho) - \frac{2}{A^3\rho^2}\,K_1(2A\rho).$$

Here $I_0(\rho)$ and $I_1(\rho)$ are the modified Bessel functions of the first kind [11, 12], which have the following representations as $\rho \to \infty$:

$$I_m(\rho) = \sum_{k=0}^{\infty} \frac{(\rho/2)^{2k+m}}{\Gamma(k+1)\Gamma(m+k+1)},$$

and $K_0(\rho)$ and $K_1(\rho)$ are the modified Bessel functions of the second kind or the Macdonald functions which tend to zero as $\rho \to \infty$:

$$K_m(\rho) = (-1)^{m+1}I_m(\rho)\Big(\ln\frac{\rho}{2}+C\Big) + \frac{1}{2}\sum_{k=0}^{m-1}(-1)^k\,\frac{(m-k-1)!}{k!}\Big(\frac{\rho}{2}\Big)^{2k-m} + \frac{(-1)^m}{2}\sum_{k=0}^{\infty}\frac{(\rho/2)^{2k+m}}{k!(m+k)!}\Big(\sum_{s=1}^{k}\frac{1}{s}+\sum_{s=1}^{k+m}\frac{1}{s}\Big)$$

(m is an integer and $C = 0.5772\ldots$ is the Euler constant).

Using the boundary conditions (1.8), we obtain the system of linear algebraic equations $A\{C_1,\dots,C_8\}^{\mathrm{t}} = \{0,0,0,0,p/2,p/2,-p/2,0\}^{\mathrm{t}}$ for the constants $C_1,\dots,C_8$ ($R_1\to\infty$), where

$$A=\begin{bmatrix} \dfrac{4}{æ-1} & -\dfrac{2}{R_0^2} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\dfrac{6}{R_0^4} & -\dfrac{8}{(æ+1)R_0^2} & 2 & 0 & S^{(7)}_{\rho\rho}(L) & S^{(8)}_{\rho\rho}(L)\\ 0 & 0 & -\dfrac{6}{R_0^4} & -\dfrac{4}{(æ+1)R_0^2} & -2 & -\dfrac{12R_0^2}{3-æ} & S^{(7)}_{\rho\varphi}(L) & S^{(8)}_{\rho\varphi}(L)\\ 0 & 0 & 0 & -\dfrac{2}{A^2BR_0^3} & 0 & -\dfrac{6(æ+1)R_0^2}{(3-æ)A^2B} & M^{(7)}_{\rho z}(L) & M^{(8)}_{\rho z}(L)\\ \dfrac{4}{æ-1} & -\dfrac{2}{R_1^2} & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -\dfrac{6}{R_1^4} & -\dfrac{8}{(æ+1)R_1^2} & 2 & 0 & S^{(7)}_{\rho\rho}(2AR_1) & S^{(8)}_{\rho\rho}(2AR_1)\\ 0 & 0 & -\dfrac{6}{R_1^4} & -\dfrac{4}{(æ+1)R_1^2} & -2 & -\dfrac{12R_1^2}{3-æ} & S^{(7)}_{\rho\varphi}(2AR_1) & S^{(8)}_{\rho\varphi}(2AR_1)\\ 0 & 0 & 0 & -\dfrac{2}{A^2BR_1^3} & 0 & -\dfrac{6(æ+1)R_1^2}{(3-æ)A^2B} & M^{(7)}_{\rho z}(2AR_1) & M^{(8)}_{\rho z}(2AR_1) \end{bmatrix}.$$

The solution of this system has the form

$$C_1=\frac{p(æ-1)}{8},\qquad C_2=\frac{pR_0^2}{4},$$

$$C_3=-\frac{pR_0^4}{4}\Big(\frac{2L(BL^2+4æ+4)K_0(L)}{L^2(2BLK_0(L)+(BL^2+4B+2æ+2)K_1(L))}+\frac{(BL^4+4BL^2+2L^2+2L^2æ+16æ+16)K_1(L)}{L^2(2BLK_0(L)+(BL^2+4B+2æ+2)K_1(L))}\Big), \quad (2.7)$$

$$C_4=\frac{pR_0^2(æ+1)}{4}\,\frac{B(2LK_0(L)+(4+L^2)K_1(L))}{2BLK_0(L)+(BL^2+4B+2æ+2)K_1(L)},\qquad C_5=\frac{p}{4},$$

$$C_6=0,\qquad C_7=0,\quad C_8=\frac{p(æ+1)L}{2(2BLK_0(L)+(BL^2+4B+2æ+2)K_1(L))}.$$

Here the dimensionless quantity $L=2AR_0$ is introduced for brevity.

The solution of system (2.2) corresponding to the Cosserat pseudomedium is also determined by relations (2.5)–(2.7). In this case, however, $B=1$ (2.4), which corresponds to the limiting case as $\alpha\to\infty$.

The displacement vector and the components of the stress tensor obtained within the framework of the symmetric theory of elasticity and the rotation vector determined by relation (1) have the form

$$u^*_\rho(\rho,\varphi)=C_1^*\rho+\frac{C_2^*}{\rho}+\Big(\frac{C_3^*}{\rho^3}+\frac{C_4^*}{\rho}+C_5^*\rho\Big)\cos2\varphi,$$

$$u^*_\varphi(\rho,\varphi)=\Big(\frac{C_3^*}{\rho^3}-C_4^*\,\frac{æ-1}{(æ+1)\rho}-C_5^*\rho\Big)\sin2\varphi,\qquad \omega^*_z(\rho,\varphi)=\frac{C_4^*}{\rho^2}\sin2\varphi,$$

$$\sigma^*_{\rho\rho}(\rho,\varphi)=\frac{4C_1^*}{æ-1}-\frac{2C_2^*}{\rho^2}+\Big(-C_3^*\,\frac{6}{\rho^4}-\frac{8C_4^*}{(æ+1)\rho^2}+2C_5^*\Big)\cos2\varphi, \quad (2.8)$$

$$\sigma^*_{\rho\varphi}(\rho,\varphi)=\Big(-C_3^*\,\frac{6}{\rho^4}-C_4^*\,\frac{4}{(æ+1)\rho^2}-2C_5^*\Big)\sin2\varphi,\quad \sigma^*_{\varphi\rho}(\rho,\varphi)=\sigma^*_{\rho\varphi}(\rho,\varphi),$$

$$\sigma^*_{\varphi\varphi}(\rho,\varphi)=C_1^*\,\frac{4}{æ-1}+\frac{2C_2^*}{\rho^2}+\Big(C_3^*\,\frac{6}{\rho^4}-2C_5^*\Big)\cos2\varphi,\quad D^*=\Big|\frac{u^*_\rho(R_0,0)}{u^*_\rho(R_0,\pi/2)}\Big|,$$

where $C_1^*=p(æ-1)/8$, $C_2^*=pR_0^2/4$, $C_3^*=-pR_0^4/4$, $C_4^*=pR_0^2(æ+1)/4$, $C_5^*=p/4$, and $C_6^*=0$ [7].

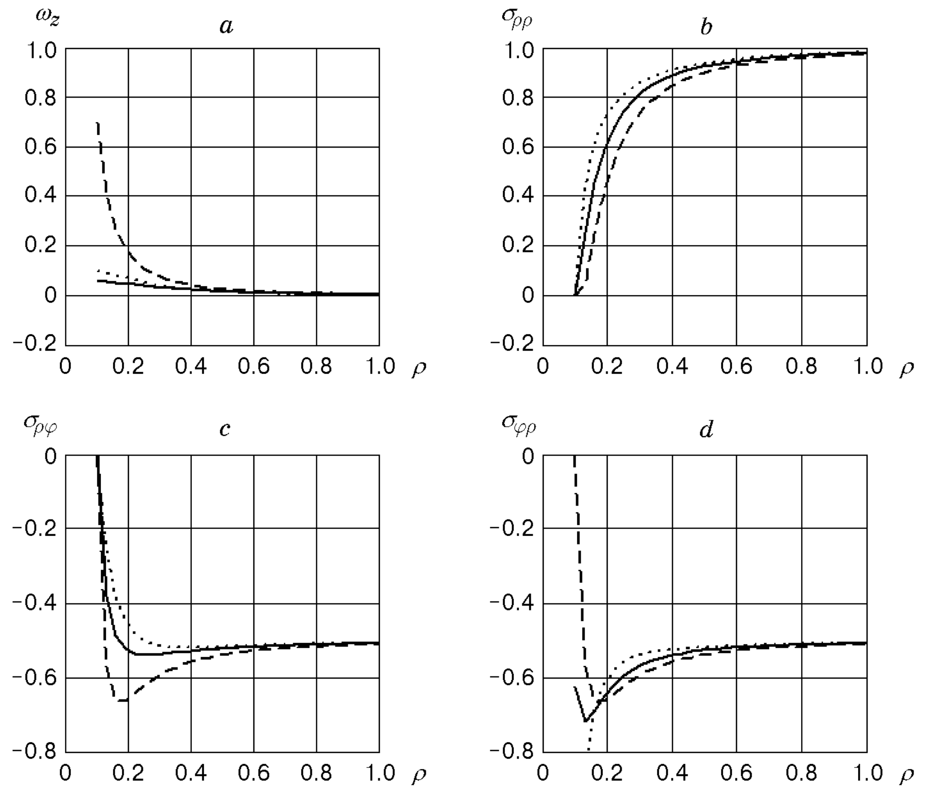
a
ω_z
1.0
0.8
0.6
0.4
0.2
0
−0.2
0
0.2
0.4
0.6
0.8
1.0
ρ
b
$\sigma_{\rho\rho}$
c
$\sigma_{\rho\varphi}$
−0.4
−0.6
−0.8
d
$\sigma_{\varphi\rho}$

Fig. 2

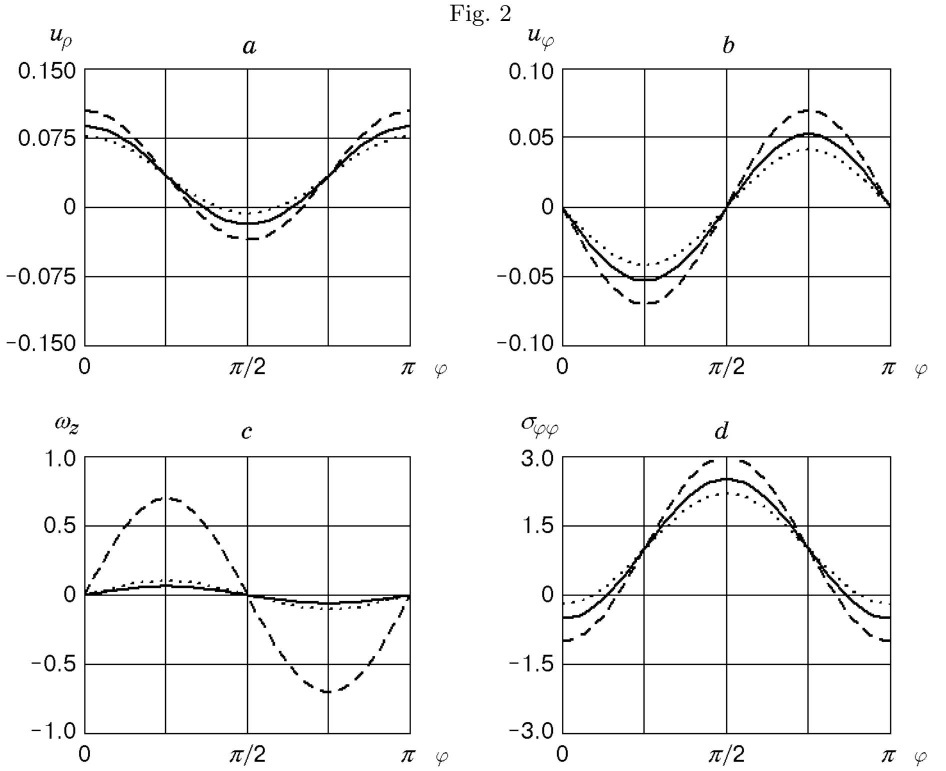
a
u_ρ
0.150
0.075
0
−0.075
−0.150
$\pi/2$
π
φ
b
u_φ
0.10
0.05
−0.05
−0.10
c
ω_z
1.0
0.5
−0.5
−1.0
d
$\sigma_{\varphi\varphi}$
3.0
1.5
−1.5
−3.0

Fig. 3

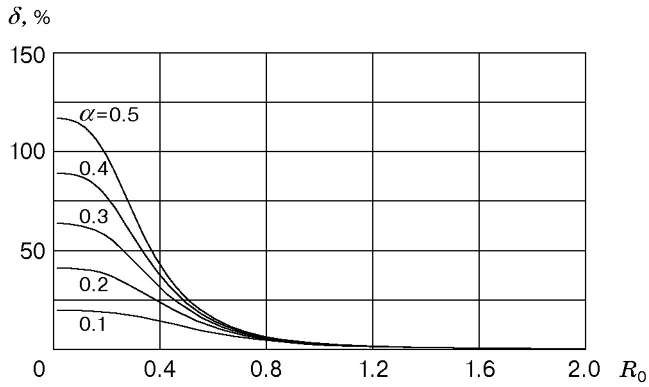

Fig. 4

3. Parametric Analysis of the Solution. Using the above solutions, we compare the stress–strain states in the neighborhood of a circular hole, obtained for the Cosserat medium, Cosserat pseudomedium, and symmetric medium.

Figure 2 shows the rotation-vector component ω_z versus the coordinate ρ for $\varphi = \pi/4$ (Fig. 2a), the stress-tensor component $\sigma_{\rho\rho}$ versus the coordinate ρ for $\varphi = 0$ (Fig. 2b), the stress-tensor component $\sigma_{\rho\varphi}$ versus the coordinate ρ for $\varphi = \pi/4$ (Fig. 2c), and the stress-tensor component $\sigma_{\varphi\rho}$ versus the coordinate ρ for $\varphi = \pi/4$ (Fig. 2d). The solid curves refer to the nonsymmetric medium, the dashed curves to the symmetric medium, and the dotted curves to the Cosserat pseudomedium. These diagrams are obtained for the physical constants $\alpha = 0.5$, $\gamma = \varepsilon = 1$, and $æ = 1.8$ and the radius of the internal circle $R_0 = 0.1$.

Figure 3a–d shows the radial component u_ρ and azimuthal component u_φ of the displacement vector, the rotation-vector component ω_z, and the stress-tensor component $\sigma_{\varphi\varphi}$, respectively, as functions of the coordinate φ for $\rho = R_0$ (notation the same as in Fig. 2).

It follows from the graphs shown in Fig. 3a that one can use the parameter D characterizing the distortion of the contour of a circular hole as an experimentally measured macroquantity.

To estimate the discrepancy between solution (2.5) obtained within the framework of the nonsymmetric theory and the classical solution (2.8), we introduce the quantity $\delta = |(D - D^*)/D^*| \cdot 100\,\%$. Figure 4 shows δ as a function of the hole radius R_0 for various α. One can see that the effect of the moment description of the behavior of the material on the quantity δ becomes pronounced as the characteristic dimension (radius of the circular hole) decreases. The reason is that the dimensionless moment solution depends on the characteristic dimension, whereas the classical solution does not.

Conclusions. The qualitative and numerical analysis of the analytical solutions considered above and dependences plotted in Figs. 2–4 leads to the following conclusions.

Nondimensionalization of the resulting analytical solutions shows that the dimensionless moment solution depends on the characteristic dimension, and the classical solution does not.

As the size of the circular hole is diminished, the discrepancy between the macroquantities obtained within the framework of the nonsymmetric theory and classical theories becomes more pronounced (Fig. 4).

As an experimentally measured quantity, one can use the parameter D that characterizes the distortion of the circular hole.

The discrepancies between the classical and nonsymmetric solutions and the solution obtained for the Cosserat pseudomedium depend on the material constants. The classical solution and the solution for the Cosserat pseudomedium are the limiting cases of the nonsymmetric solution as $\alpha \to 0$ and $\alpha \to \infty$, respectively. The discrepancy between the classical solution and the solution for the Cosserat pseudomedium is determined by the quantity $\gamma + \varepsilon$.

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